

## Thermodynamical cost of accessing quantum information

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2005 J. Phys. A: Math. Gen. 38 7175

(<http://iopscience.iop.org/0305-4470/38/32/007>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.92

The article was downloaded on 03/06/2010 at 03:52

Please note that [terms and conditions apply](#).

# Thermodynamical cost of accessing quantum information

Koji Maruyama<sup>1</sup>, Āaslav Brukner<sup>2</sup> and Vlatko Vedral<sup>2,3</sup>

<sup>1</sup> Laboratoire d'Information Quantique and QUIC, CP 165/59, Université Libre de Bruxelles, Avenue F D Roosevelt 50, 1050 Bruxelles, Belgium

<sup>2</sup> Institut für Experimentalphysik, Universität Wien, Boltzmannngasse 5, A-1090 Wien, Austria

<sup>3</sup> The School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, UK

Received 27 April 2005

Published 27 July 2005

Online at [stacks.iop.org/JPhysA/38/7175](http://stacks.iop.org/JPhysA/38/7175)

## Abstract

Thermodynamics is a macroscopic physical theory whose two very general laws are independent of any underlying dynamical laws and structures. Nevertheless, its generality enables us to understand a broad spectrum of phenomena in physics, information science and biology. Does thermodynamics then imply any results in quantum information theory? Taking accessible information in a system as an example, we show that thermodynamics implies a weaker bound on it than the quantum mechanical one (the Holevo bound). In other words, if any post-quantum physics should allow more information storage it could still be under the umbrella of thermodynamics.

PACS numbers: 03.65.Ta, 03.67.–a, 05.70.–a

## 1. Introduction

Since ‘information is physical’ [1] the performance of information theoretic tasks is ultimately governed by the underlying physical laws used to process it. For example, in quantum mechanics, information that can be stored or accessed is limited by the Holevo bound [2]. On the other hand, information theory is also deeply connected with thermodynamics as most notably demonstrated by the resolution of the long-standing Maxwell’s demon paradox [3, 4] on the basis of Landauer’s erasure principle [5]. The insight acquired from Landauer’s principle enabled the demon paradox to be extended to the quantum regime [4, 6–8] and its link with limits on efficiency of certain quantum information processing [9] has also been established [10, 11]. The amount of heat convertible into work in reversible and irreversible processes was considered in the context of quantum distinguishability [12] and it was shown that distinguishing non-orthogonal states perfectly would lead to the violation of the second law of thermodynamics [13]. The latter suggests that altering quantum laws may have dramatic consequences on our other theories.

Here we derive a thermodynamical bound on accessible information in quantum mechanics from the second law of thermodynamics, which states in Kelvin's form 'There is no thermodynamical cycle whose sole effect is the conversion of heat withdrawn from a reservoir into mechanical work'. The background of our motivation is the fact that the generality of thermodynamical laws has led physicists to derive many, at first sight unrelated, results, such as general relativity [14], the superposition principle in quantum mechanics [15] and the wave nature of light [16] to name a few. In this paper, we investigate what constraint the second law imposes on accessible information and show that thermodynamical bound is *weaker* than the Holevo bound.

Assumptions we make here are (a) entropy: the von Neumann entropy is equivalent to the thermodynamical entropy, (b) statics and measurement: a physical state is described by a 'density' matrix, and the state after a measurement is a new state that corresponds to the outcome ('projection postulate'), (c) dynamics: there exist isentropic transformations. These rules can also describe probability distributions in classical phase space. Although we will use Dirac's ket notation for convenience, this does not mean that we use the full machinery of the Hilbert spaces (such as the notion of inner product) and we never use the Born trace rule for calculating probabilities.

By accessible information we mean information obtained from an arbitrary measurement on a given system. To give a precise form of the Holevo bound let us consider two protagonists, Alice and Bob. Suppose Alice has a classical information source preparing symbols  $i = 1, \dots, n$  with probabilities  $p_1, \dots, p_n$ . Bob attempts to determine the actual preparation  $i$  as best as he can. Thus, after Alice prepared a state  $\rho_i$  with probability  $p_i$  she gives the state to Bob, who makes a general quantum measurement (*positive operator valued measure* or *POVM*) with elements  $E_j = E_1, \dots, E_m, \sum_{j=1}^m E_j = \mathbf{I}$ , on that state. On the basis of the measurement result he infers Alice's preparation  $i$ . The Holevo bound is an upper bound on accessible information, i.e.

$$I(A : B) \leq S(\rho) - \sum_i p_i S(\rho_i), \quad (1)$$

where  $I(A : B)$  is the mutual information between the set of Alice's preparations  $i$  and Bob's measurement outcomes  $j$ ,  $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$  is the von Neumann entropy and  $\rho = \sum_i p_i \rho_i$ . The equality in expression (1) is achieved if all  $\rho_i$  mutually commute, that is,  $[\rho_i, \rho_j] = 0$  for all  $i, j$ , and the measurement is performed in the joint eigenbasis of  $\rho_i$ . We will refer to this case as 'classical', as it corresponds to distinguishing between classical probability distributions.

## 2. The thermodynamical cycle

Let us now derive the thermodynamical bound on the mutual information  $I(A : B)$ . To this end we will consider a thermodynamical loop that involves a conversion of heat into work, whose amount is equal to  $kT \ln 2 I(A : B)$  (throughout the paper we call the  $kT \ln 2$  unit of work as one bit. Here  $k$  is the Boltzmann constant and  $T$  is the temperature). Examining the condition on the work balance imposed by the second law, we will have a thermodynamical bound on  $I(A : B)$ .

Consider a vessel of volume  $V$  filled with  $N$  molecules of dilute, inert, ideal gas. Suppose that  $p_1 N$  molecules occupy the space on the left-hand side ( $L$ ) of the vessel, whose volume is  $p_1 V$ , and each individual molecule is in the quantum state  $|\psi_1\rangle$ . Similarly,  $p_2 N$  molecules ( $p_1 + p_2 = 1$ ) are in the right-hand side ( $R$ ) of volume  $p_2 V$  and are all in  $|\psi_2\rangle$ . The two states,  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , can be thought of as the states of an internal degree of freedom such as spin.

The two types of molecules are initially separated by a partition and the pressures on both sides are equal. Note that this situation differs from the encoding/decoding scenario given above in which Bob has no access to the spatial degree of freedom but can only measure the internal degree of freedom of the molecules. Even though we primarily deal with only two pure quantum states and projective measurements with two possible outcomes, our consideration can easily be generalized to arbitrary numbers of general states and measurement outcomes.

We can now have a thermodynamical loop formed by two different paths between the above initial thermodynamical state to the final state. In the final state, both constituents,  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , will be distributed uniformly over the whole volume of the vessel. Hence, each molecule in the final state can be described by  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , regardless of the position in the vessel. One of the paths converts heat into work, involving measurement (and thus it is irreversible, in general), while the other path, consisting of a quasi-static reversible process and isentropic transformations, requires some work consumption.

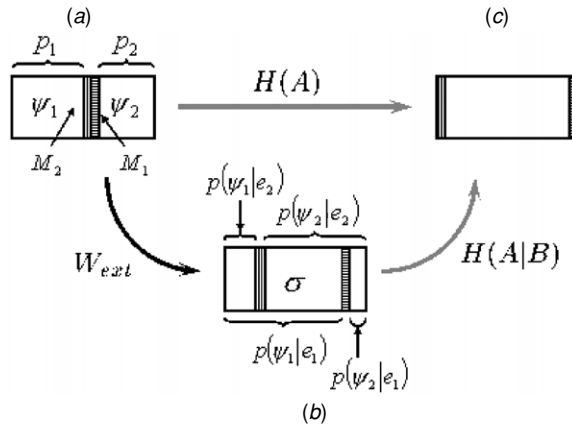
### 2.1. Work-extracting process

The work-extracting process proceeds as follows. Suppose that we have two semipermeable membranes,  $M_1$  and  $M_2$ , which separate two perfectly distinguishable (orthogonal) states  $|e_1\rangle$  and  $|e_2\rangle$  ( $=|e_1^\perp\rangle$ ). These membranes were considered by von Neumann [17] and Peres [13] and shown to be physically legitimate. The membrane  $M_1$  acts as a completely opaque wall to molecules in  $|e_1\rangle$ , but it is transparent to molecules in  $|e_2\rangle$ . Similarly,  $M_2$  is opaque to molecules in  $|e_2\rangle$  and transparent to  $|e_1\rangle$ . Thus, for example, a state  $|\psi_i\rangle$  is reflected by  $M_1$  to become  $|e_1\rangle$  with (conditional) probability  $p(e_1|\psi_i)$  and goes through with probability  $p(e_2|\psi_i)$ , being projected onto  $|e_2\rangle$ . This corresponds to the quantum (projective) measurement on molecules in the basis  $\{|e_1\rangle, |e_2\rangle\}$ , however, we do not compute these probabilities specifically as stated above.

We replace the partition separating  $|\psi_1\rangle$  and  $|\psi_2\rangle$  with the two membranes,  $M_1$  and  $M_2$ . Keeping its temperature constant by contact with a heat bath of temperature  $T$ , each gas of molecules in  $|e_1\rangle$  or  $|e_2\rangle$  can expand isothermally until the pressures of each gas component on both sides of a membrane become equal. The amount of the mechanical work  $W_{\text{ext}}$ , which can be withdrawn from the heat bath, is equal to the accessible information  $I(A : B)$ , which is the amount of information Bob can obtain about Alice's preparation by measurement on  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$  in the basis  $\{|e_1\rangle, |e_2\rangle\}$ . Figure 1 shows the equivalence between  $W_{\text{ext}}$  and  $I(A : B)$ . The same correspondence exists also if the initial state is a collection of mixed states, such as  $\{p_i, \rho_i\}$ , which means that Alice provides  $\rho_i$  with probability  $p_i$ . The transformation from the post-work-extraction state, which we call  $\sigma$  hereafter, to the final state  $\rho$  can be done by a process shown in figure 2 and the minimum work needed is given by  $\Delta S = S(\sigma) - S(\rho)$ .

### 2.2. Maxwell's demon as membranes

There is an alternative way to look at the process with semipermeable membranes. Maxwell's demon, who is sitting somewhere in (or by) the box, measures the state of each molecule in the basis  $\{|e_1\rangle, |e_2\rangle\}$  and memorizes all outcomes from the measurement on all molecules. Depending upon the actual outcome he operates a membrane  $M_i$  (i.e. 'controls tiny doors' on  $M_i$ ) so that only  $|e_i\rangle$  is reflected and the other state can go through. One may then ask whether or not some work needs to be consumed to erase the information recorded in his memory at a certain stage to close the thermodynamical cycle. We show that it is not necessary. Let  $|m\rangle$  denote the demon's memory for outcome  $m$ . After the observation and work-extraction by demon, the joint system of the principal system  $P$  and the memory  $M$ ,  $\sigma^{PM}$ , can be described

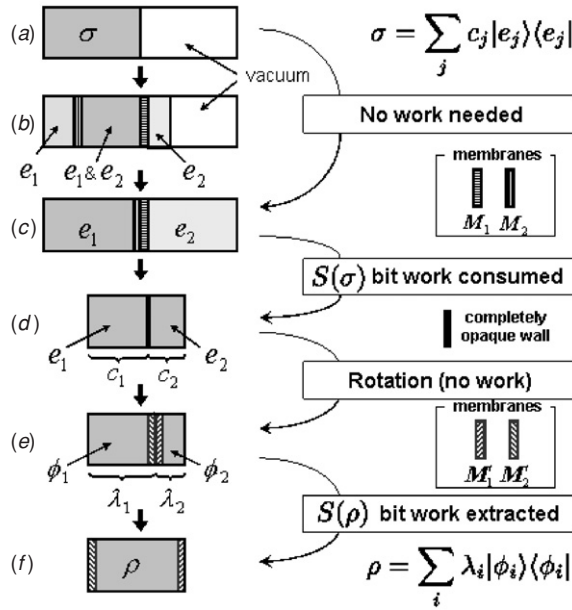


**Figure 1.** Equivalence between the extractable work  $W_{\text{ext}}$  and the accessible information  $I(A : B)$ . Suppose that each  $|\psi_i\rangle$  in the initial preparation was correlated with a state of another degree of freedom so that in the initial state (a) there are  $|\psi_1\rangle|L\rangle$  on the left of the vessel and  $|\psi_2\rangle|R\rangle$  on the right. As this auxiliary system is hypothetical, we cannot access this degree of freedom throughout the process discussed in the main text. Nevertheless, if we could make use of membranes that distinguish  $|L\rangle$  and  $|R\rangle$ , then we can extract  $H(A)$  bits of work to reach the state (c), where  $H(A) = H(p_i) = -\sum_i p_i \log_2 p_i$  is the Shannon entropy. If we use the ‘proper’ membranes,  $M_1$  and  $M_2$ , that measure  $|\psi_i\rangle$  in a basis  $\{|e_i\rangle\}$ ,  $W_{\text{ext}}$  bits of work will be extracted, and if  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are not perfectly distinguishable the membranes will stop before reaching the end of vessel (as in (b)). Let us consider the gas in  $|e_1\rangle$ , for example. In (b), the numbers of  $|e_1\rangle$ -molecules on the left-hand and the right-hand sides of  $M_1$  are  $p(\psi_1)p(e_1|\psi_1)N = p(e_1)p(\psi_1|e_1)N$  and  $p(e_1)p(\psi_2|e_1)N$ , respectively, where  $p(x)$  is the proportion of  $|x\rangle$  to  $N$  (thus  $p(\psi_i) = p_i$ ), and  $p(x|y)$  represents the probability of finding  $|x\rangle$  in  $|y\rangle$ . By using the same membranes as those used in the direct path from (a) to (c) (namely, distinguishing  $|e_1\rangle|L\rangle$  and  $|e_1\rangle|R\rangle$  that are separated by  $M_1$ ),  $H(A|B)$  bits of work can be extracted in the process from (b) to (c). As these hypothetical work-extraction processes with the auxiliary system are quasi-static and reversible, a simple relation,  $H(A) = W_{\text{ext}} + H(A|B)$ , holds and this means  $W_{\text{ext}} = I(A : B)$ .

as  $\sigma^{PM} = \sum_m P_m \rho P_m \otimes |m\rangle\langle m|$ , where  $P_m = |e_m\rangle\langle e_m|$  are projection operators. Hence, unlike the usual discussion of the erasure principle with Szilard’s engine [18], erasing the demon’s memory in this case is a logically reversible process [5] due to a perfect correlation between  $P$  and  $M$ . Therefore a controlled-NOT-like global (isentropic) operation between  $P$  and  $M$  can reset the state of  $M$  to a standard initial state without consuming work. We can consider that isentropic transformations, in principle, involve no work transfer [17]. The difference from the memory erasure in the Szilard model is that the degrees of freedom used for the work-extraction and measurement are, in our case, not the same. Here the external (spatial) and the internal (spin) ones are used, while only the external one is employed in the Szilard model.

### 2.3. The returning path

Another path, which is reversible, from the initial state to the final state is as follows. Let  $\{|\phi_1\rangle, |\phi_2\rangle\}$  be an orthogonal basis which diagonalizes the density matrix  $\rho$ , such that  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$ , where  $\lambda_k$  are eigenvalues of  $\rho$ . Since any of  $\{|\psi_1\rangle, |\psi_2\rangle\}$  and  $\{|\phi_1\rangle, |\phi_2\rangle\}$  is a pure state, appropriate isentropic transformations can transform the initial state to a state in which  $\lambda_1 V$  of the vessel on the left is occupied with  $|\phi_1\rangle$  and  $\lambda_2 V$  on the right with  $|\phi_2\rangle$ . By using new semipermeable membranes  $M'_1$  and  $M'_2$ , which distinguish  $|\phi_1\rangle$  and  $|\phi_2\rangle$  perfectly, we obtain a state  $\rho$  uniformly distributed over the volume  $V$ , after gaining



**Figure 2.** The thermodynamical process to transform  $\sigma$  into  $\rho$ . Firstly, after attaching an empty vessel of the same volume to that containing the gas  $\sigma$ , the membranes  $M_j$  are used to separate two orthogonal states  $|e_1\rangle$  and  $|e_2\rangle$  ((a) to (c)). As the distance between the movable opaque wall and the membrane  $M_2$  is kept constant, this process entails no work consumption/extraction. As  $\sigma = \sum_j c_j |e_j\rangle \langle e_j|$ , compressing each  $|e_j\rangle$ -gas into the volume of  $c_j V$  as in (d) makes the pressures of gases equal and this compression requires  $S(\sigma) = -\sum_j c_j \log_2 c_j$  bits of work. Secondly, quantum states of gases are isentropically transformed, thus without consuming work, so that the resulting state (e) will have  $\lambda_j N$  molecules in  $|\phi_j\rangle$ , where  $\rho = \sum_j \lambda_j |\phi_j\rangle \langle \phi_j|$  is the eigendecomposition of  $\rho$ . To reach (f),  $S(\rho)$  bits of work can be extracted by using membranes that distinguish  $|\phi_j\rangle$ . As a result, the work needed for the transformation  $\sigma \rightarrow \rho$  is  $S(\sigma) - S(\rho)$  bits.

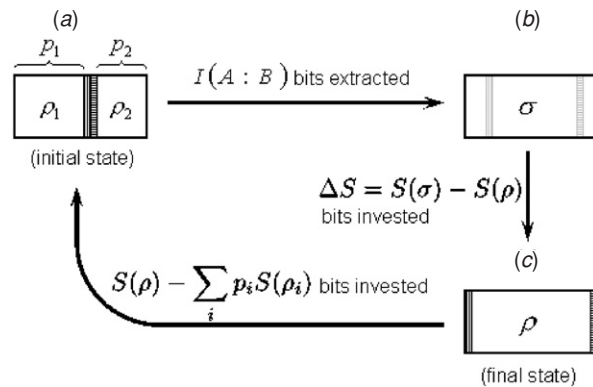
$S(\rho)$  bits of work. As this transformation from  $\{p_i, |\psi_i\rangle\}$  to  $\rho$  via  $\{\lambda_i, |\phi_i\rangle\}$  can be carried out reversibly, the initial state can be restored from  $\rho$  by consuming  $S(\rho)$  bits of work.

If the initial state is a combination of mixed states with corresponding weights as  $\{p_i, \rho_i\}$ , the extractable work to reach  $\rho = \sum_i p_i \rho_i$  becomes  $S(\rho) - \sum_i p_i S(\rho_i)$ . This can be seen by considering a process  $\{p_i, \rho_i\} \xrightarrow{(i)} \{p_i \mu_j^i, |\omega_j^i\rangle\} \xrightarrow{(ii)} \{\lambda_k, |\phi_k\rangle\} \xrightarrow{(iii)} \rho$ , where  $\{\mu_j^i, |\omega_j^i\rangle\}$  and  $\{\lambda_k, |\phi_k\rangle\}$  are the sets of eigenvalues and eigenvectors of  $\rho_i$  and  $\rho$ , respectively. The process (i) needs  $\sum_i p_i S(\rho_i)$  bits of work to be consumed, and similarly the process (iii) provides  $S(\rho)$  bits of work to us. As the process (ii) involves only isentropic transformations, nothing needs to be written in the work account book. As a result,  $S(\rho) - \sum_i p_i S(\rho_i)$  bits of work will be extracted.

### 3. The second law analysis

Now we can discuss what the second law requires for the thermodynamical loop, which proceeds as  $\{p_i, \rho_i\} \rightarrow \sigma \rightarrow \rho \rightarrow \{p_i, \rho_i\}$  (see figure 3). The second law states that the *net* work extractable from a heat bath cannot be positive after completing a cycle, i.e.  $W_{\text{ext}} - W_{\text{inv}} \leq 0$ . For the cycle described above, it can be expressed as

$$I(A : B) \leq S(\rho) - \sum_i p_i S(\rho_i) + \Delta S, \quad (2)$$



**Figure 3.** The thermodynamical cycle to discuss the second law. The cycle proceeds from the initial state (a) to the final state  $\rho$  (c) via the post-work-extraction state  $\sigma$  (b), and returns to the initial state with a reversible process. The existence of a heat reservoir at temperature  $T$  is assumed for isothermal expansion/compression processes.

where  $\Delta S = S(\sigma) - S(\rho)$ . Note that  $\sigma$  is identical to the resulting state of a projective measurement on  $\rho$  in the basis  $\{|e_1\rangle, |e_2\rangle\}$ . Thus,  $\sigma = \sum_j P_j \rho P_j$  with  $P_j = |e_j\rangle\langle e_j|$  and consequently  $\Delta S$  is always non-negative [17]. The inequality (2) holds even if the measurement by membranes was a generalized (POVM) measurement. This is because any POVM measurement on a principal system  $P$  can be realized by introducing an auxiliary system (environment)  $E$  and performing a projective measurement on  $E$  after letting  $P$  and  $E$  interact with each other under an appropriate global unitary (thus isentropic) evolution [9]. Even in this case,  $\Delta S$  can easily be shown to be non-negative by using the fact that appending a pure state to the principal system does not change the entropy.

The form of equation (2) is identical to that of equation (1) for the Holevo bound, except for an extra non-negative term,  $\Delta S$ . The existence of  $\Delta S$  is essential in the cycle, where  $I(A : B)$  bits of work is extracted, since the returning path (from (b) to (c) in figure 3) is reversible, thus optimal. This illustrates that there is a difference between the bound imposed by quantum mechanics (the Holevo bound) and the one imposed by the second law of thermodynamics. Namely, there is a region in which we could violate quantum mechanics while complying with the thermodynamical law. In the classical limit, the measurement is performed in the joint eigenbasis of mutually commuting  $\rho_i$ , consequently  $\Delta S = 0$ , and in addition the Holevo bound is saturated:  $I(A : B) = S(\rho) - \sum_i p_i S(\rho_i)$ . Thus, the classical limit and the thermodynamical treatment give the same bound.

The same saturation occurs when an appropriate collective measurement is performed on sequences of  $m$  molecules, each of which is taken from an ensemble  $\{p_i, \rho_i\}$ . When  $m$  tends to infinity  $2^{m(S(\rho) - \sum_i p_i S(\rho_i))}$  typical sequences (the sequences in which  $\rho_i$  appears about  $p_i m$  times) become mutually orthogonal and can be distinguished by ‘square-root’ or ‘pretty good’ measurements [19]. This situation is thus essentially classical, hence,  $\Delta S \rightarrow 0$  and the Holevo bound will be saturated.

An interesting implication of our result is the relationship between the second law and the erasure principle. It has been shown that the form of the Holevo bound can be obtained from the erasure principle [10]. Together with this, our result suggests that the erasure principle and the second law, which are commonly believed to be equivalent, do not necessarily give the same result in the quantum regime.

#### 4. Conclusion

As mentioned earlier in this paper, Landé has claimed that the superposition principle in quantum mechanics can be derived by purely thermodynamical arguments that are similar to our consideration [15]. If this conclusion was correct even in its spirit and if we could really derive quantum mechanics from thermodynamics, then we should also be able to confirm the Holevo bound exactly. But, as shown here this is not the case. This should, however, not necessarily be perceived as a failure of thermodynamics. It is not unlikely that quantum theory will be superseded by a higher level generalization of which it is a special case, just like classical mechanics is a limiting case of quantum mechanics. Our paper shows that even if the amount of stored information in the post-quantum theory can be greater than allowed quantum mechanically, this can happen without violating the second law.

#### Acknowledgments

KM is a Boursier de l'ULB, and is supported by the European Union through the project RESQ IST-2001-37559, the Action de Recherche Concertée de la Communauté Française de Belgique and the IUAP program of the Belgian Federal Government under grant V-18. ČB has been supported by the European Commission, Marie Curie Fellowship, Project No 500764, and the Austrian Science Foundation (FWF) Project No F1506. VV acknowledges funding from EPSRC and the European Union.

#### References

- [1] Landauer R 1991 *Phys. Today* **44** 23
- [2] Holevo A S 1973 *Problemy Pereda. Inf.* **9** 3  
Holevo A S 1973 *Probl. Inf. Trans.* **9** 177 (Engl. transl.)
- [3] Maxwell J C 1985 *Theory of Heat* (London: Longmans Green)
- [4] Leff H S and Rex A F (ed) 2003 *Maxwell's Demon 2* (Bristol: Institute of Physics Publishing)
- [5] Landauer R 1961 *IBM J. Res. Dev.* **5** 183  
Bennett C H 1982 *Int. J. Theor. Phys.* **21** 905
- [6] Zurek W H 1984 *Frontiers of Nonequilibrium Statistical Physics* ed G T Moore and M O Scully (New York: Plenum) p 151
- [7] Lubkin E 1987 *Int. J. Theor. Phys.* **26** 523
- [8] Lloyd S 1997 *Phys. Rev. A* **56** 3374
- [9] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [10] Plenio M B 1999 *Phys. Lett. A* **263** 281  
Plenio M B and Vitelli V 2001 *Contemp. Phys.* **42** 25
- [11] Vedral V 2000 *Proc. R. Soc. Lond. A* **456** 969
- [12] Lyuboshitz V L and Podgoretskii M I 1970 *Sov. Phys.—Dokl.* **194** 547  
Levitin L B 1993 *Proc. Workshop on Phys. and Comp., PhysComp'92* (Maryland: IEEE Computer Society Press) p 223
- [13] Peres A 1993 *Quantum Theory: The Concepts and Methods* (Boston, MA: Kluwer) chapter 9
- [14] Bekenstein J D 1973 *Phys. Rev. D* **7** 2333  
Jacobson T 1995 *Phys. Rev. Lett.* **75** 1260
- [15] Landé A 1952 *Phys. Rev.* **87** 267
- [16] Gabor D 1961 *Progress in Optics* ed E Wolf (Amsterdam: North-Holland) vol 1 p 111
- [17] Neumann J von 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press) chapter 5
- [18] Szilard L Z 1929 *Z. Phys.* **53** 840
- [19] Hausladen P, Jozsa R, Schumacher B, Westmoreland M and Wootters W K 1996 *Phys. Rev. A* **54** 1869  
Holevo A S 1998 *IEEE Trans. Inform. Theory* **44** 269